Lecture 4: 2.3-3.2 Lines and Planes

September 20, 2016 8:00 PM

2.7 Angles Between Vectors

Cauchy-Schwarz Inequality

$If\ u, v \in \mathbb{R}^n\ then:$

$|u \cdot v| \leq |u||v|$

Proof: $|u \cdot v|^2 = (u+v)\cdot(u+v) = u\cdot u + 2(u\cdot v) + v\cdot v$

$\leq |u|^2 + 2|u|\cdot|v| + |v|^2 = (|u|+|v|)^2$

$Hence, \underbrace{|u+v| \leq |u| + |v|}_{\text{"triangle inequality"}}$

The angle, θ, between 2 vectors is defined as:

$\cos(\theta) = \dfrac{u\cdot v}{|u|\cdot|v|}$ } based on Cauchy-Schwarz } fraction is always b/w -1 and 1

$0 \leq \theta \leq \pi$

Observe:

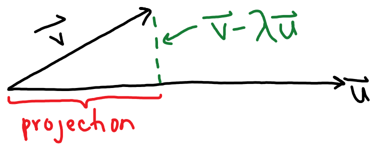
$u \perp v \iff u\, v = 0 \quad (\theta = \pi/2)$

$u \parallel v \iff \theta = 0^\circ$ or $\theta = \pi$

in which case

$u\, v = \pm |u|\, |v|$

3.3 Orthogonal projection onto lines: $u, v \in \mathbb{R}^n,\ u \neq 0$



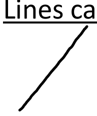
$\underbrace{proj_u v = \dfrac{v\cdot u}{u\cdot u}\cdot u}_{\text{"projection of v onto u"}}$

3.1 Describing Lines

Two many ways to describe lines

- vector form
  - $L: \{v_0 + t\cdot \mathrm{d} | t \in \mathbb{R}\}$
  - $\boldsymbol{or}\ L = [\mathrm{a,b,c}] + t[\mathrm{d}_x, \mathrm{d}_y, \mathrm{d}_z]$
- parametric form
  - $x = a + t\,\mathrm{d}_x$
  - $y = b + t\,\mathrm{d}_y$
  - $z = c + t\,\mathrm{d}_z$

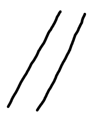
Lines can be:



identical

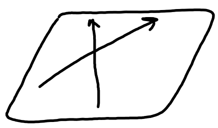
$L_1 \cap L_2 = \infty$

intersection



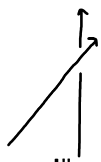
parallel

$L_1 \cap L_2$ is empty



in same plane but not parallel

$L_1 \cap L_2$ is a single point



"**skewed**"

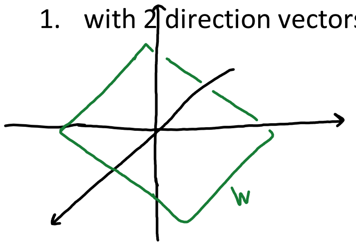
$L_1 \cap L_2$ is empty

3.2 Describing planes\

1. with 2 direction vectors and a point on the plane

3.2 Describing planes\

1. with 2 direction vectors and a point on the plane

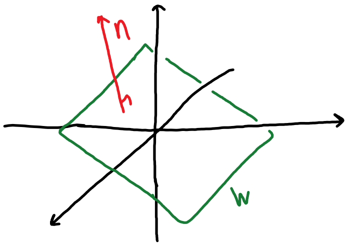


$$w = \underbrace{\{v_0 + t \cdot d_1 + s \cdot d_2 | t, s \in \mathbb{R}\}}_{\text{vector form}}$$

eg.

$$w = (2,3,1) - t(-3,4,1) + s(7,1,-4)$$

2. with normal vector orthogonal to the plane



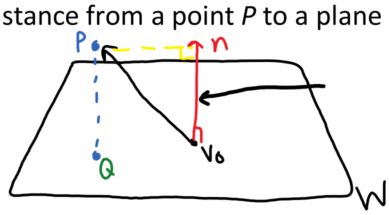
$$w = \{\text{v} \in \mathbb{R} | (v - v_0) \cdot n = 0\}$$
$$= \underbrace{\{\text{v} \in \mathbb{R} | (v \cdot n = v_0 \cdot n)\}}_{\text{cartesian form}}$$

eg.

$$2x - y + z = 16$$

3.3 Working with normal vectors

a) distance from a point *P* to a plane *w*:



Distance from *P* to *w*:

$= \|P - Q\|$

$= \|proj_n(P - v_0)\|$

b) angle between 2 planes is the angle between their normal vectors
ie. if n_1=[-2, 4, 3] and n_2=[1,4,5] then:

$$\cos(\theta) = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

3.4 The cross product

Given 2 vectors on the same plane, we can determine a vector that is perpendicular to them both using the cross product.

$v \text{ x } w =$

Properties of the cross product:

a) $u \times v = -(v \times u)$

b) $(u \times v)\ u = (u \times v)\ v$

c) $(u + v) \times w = u \times w + v \times w$

d) $\|u \times w\| = \|u\|\ \|w\| \sin(\vartheta)$

→ area of the parallelogram spanned by u and w

More applications:

- area of a triangle given three vertices A, B, and C
  $$= \frac{|\overrightarrow{AB} \text{ x } \overrightarrow{AC}|}{2}$$
- volume of a parallelepiped formed by vectors *u*, *v*, and *w*
  $$= |u \cdot (v \text{ x } w)|$$